



Laminar flow and heat transfer in the boundary-layer of non-Newtonian fluids over a stretching flat sheet

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ABSTRACT

A theoretical analysis of the laminar boundary-layer flow and heat transfer of power-law non-Newtonian fluids over a stretching sheet with the sheet velocity distribution of the form $U_w = Cx^m$ and the wall temperature distribution of the form $T_w = T_\infty + Ax^\gamma$ is presented, where x denotes the distance from the slit from which the surface emerges and C and A are constants, m and γ denote, the sheet velocity exponent and the temperature exponent, respectively. Within the framework of the boundary layer approximations, the nonlinear boundary layer momentum equation and the energy equation are reduced to a set of ordinary differential equations. It is found that when the velocity exponent $m = 1/3$ or the power-law index $n = 1$, the similarity solutions are in existence for both the momentum equation and the energy equation. Analytical approximations with high accuracy for the reduced velocity and temperature profiles are obtained using a new procedure based on the homotopy analysis method. Besides, the effects of the parameters m , n and the Prandtl number Pr on the flow are investigated.

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1. Introduction

Investigations of boundary layer flow and heat transfer of viscous fluids over a flat sheet are important in many manufacturing processes, such as polymer extrusion, drawing of copper wires, continuous stretching of plastic films and artificial fibers, hot rolling, wire drawing, glass-fiber, metal extrusion, and metal spinning. Among these studies, Sakiadis [1] initiated the study of the boundary layer flow over a stretched surface moving with a constant velocity and formulated a boundary-layer equation for two-dimensional and axisymmetric flows. Tsou et al. [2] analyzed the effect of heat transfer in the boundary layer on a continuous moving surface with a constant velocity and experimentally confirmed the numerical results of Sakiadis [1]. Erickson et al. [3] extended the work of Sakiadis [1] to include blowing or suction at the stretched sheet surface on a continuous solid surface under constant speed and investigated its effects on the heat and mass transfer in the boundary layer. The related problems of a stretched sheet with a linear velocity and different thermal boundary conditions in Newtonian fluids have been studied, theoretically, numerically and experimentally, by many researchers, such as Crane [4], Fang [5–8], Fang and Lee [9].

Many industrial fluids such as particulate slurries (china clay and coal in water, sewage sludge, etc.), multi-phase mixtures (oil–water emulsions, gas–liquid dispersions, froths and foams, butter), pharmaceutical formulations, cosmetics and toiletries, paints, synthetic lubricants, biological fluids (blood, synovial fluid, saliva), and foodstuffs (jams, jellies, soups, marmalades) in their flow characteristics can be sorted into the non-Newtonian fluids. These kinds of fluids possess two distinguishable properties. One is that their viscosity obeys a relation analogous to the Hookian relation between a stress and its strain, i.e. the stress is proportional to the strain rate. The other is that their Reynolds number and Grashof number

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are low owing to their large *apparent viscosities*. In engineering practice, the laminar flows of non-Newtonian fluids are encountered more common than those Newtonian fluid flows. Among those most popular models for the explanation of the non-Newtonian behavior of fluids, the empirical Ostwald-de Waele model (or the power-law model) gains much acceptance due to its apparent simplicity in mathematics and its adequate description of many non-Newtonian fluids over the most important range of shear rates.

Studies of non-Newtonian fluids flow over moving sheets have been received considerable attention in the past decades due to their important applications in engineering and science. Among these works, the theoretical analysis of boundary-layer equations for non-Newtonian power-law fluids was first performed by Schowalter [10] and Acrivos et al. [11]. The boundary-layer equations for a pseudo-plastic fluid were formulated, and the conditions for the existence of the similarity solutions were given by Schowalter [10]. A similarity solution to the boundary-layer flow of a power-law non-Newtonian fluid over a horizontal flat plate was obtained in [11]. Wu and Thompson [12] experimentally found that even if the Reynolds number is not large, the boundary-layer equations are still valid and accurate for the flow of shear-thinning fluids. Denier and Dabrowski [13] investigated the boundary-layer equations describing the steady uniform flow of a non-Newtonian fluid past a flat plate. They found that: (i) for shear-thinning fluids ($n < 1$) the flow in inner boundary layer admits a solution that decays algebraically into the far field, the outer boundary layer allows the algebraically decaying solutions to be matched smoothly with the solutions in the outer potential flow; (ii) for shear-thickening fluids ($n > 1$), the inner-boundary-layer solutions have a finite-width crisis which leads to that a secondary viscous adjustment layer is required to ensure correct matching with the far-field boundary conditions. Andersson and Kumaran [14] made an analysis on the non-Newtonian fluid flow over a moving plane sheet with its surface velocity proportional to the distance from the slit raised to an arbitrary power m . Their numerical results showed that the boundary layer thickness decreases monotonically with increasing values of power-law index n , which is totally different from the conclusion given by Hassanien et al. [15]. They further obtained a four-term series expansion which provides an excellent agreement to the numerical solutions over the range $0.5 \leq n \leq 1.5$. Chen [16] analyzed the boundary-layer flow and heat transfer of a non-Newtonian power-law fluid over a flat sheet moving with a power-law velocity in the presence of a transverse magnetic field.

The aim of this paper is to analyze the boundary-layer flow and heat transfer of power-law non-Newtonian fluids over a stretching sheet with the sheet velocity distribution of the form $U_w = Cx^m$ and the wall temperature distribution of the form $T_w = T_\infty + Ax^\gamma$. The similarity equations will be formulated and the existence of the similarity solutions will be discussed. Then the homotopy analysis method [17–21] will be applied to give accurate analytical approximations for this flow problem.

2. Basic equations

Consider the steady, two-dimensional laminar flow and heat transfer of an incompressible, viscous non-Newtonian power-law fluid over a stretching flat sheet. It is assumed that the sheet moves with a velocity according to a power-law form, i.e. $U_w = Cx^m$, and is subject to a prescribed surface temperature, i.e. $T_w = T_\infty + Ax^\gamma$. It is also assumed that the viscous dissipation is neglected. With these assumptions and invoking the boundary layer approximations, the governing equations for the boundary layer flow and heat transfer are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial \tau_{xy}}{\partial y}, \quad (2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}, \quad (3)$$

subject to the boundary conditions

$$u(x, 0) = U_w = Cx^m, \quad v(x, 0) = 0, \quad T(x, 0) = T_\infty + Ax^\gamma, \quad (4a)$$

$$u \rightarrow 0, T \rightarrow T_\infty \quad \text{as } y \rightarrow \infty, \quad (4b)$$

where ρ is the density of the fluid, α is the thermal diffusivity of the fluid, the shear stress tensor is defined by the Ostwald-de-Wäle model, see Andersson et al. [22] or Liao [18],

$$\tau_{ij} = 2K(2D_{kl}D_{kl})^{(n-1)/2}D_{ij} \quad (5)$$

in which

$$D_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (6)$$

denotes the rate of stretching tensor, K is the consistency coefficient, and n is the index in the power-law variation of the shear stress of a non-Newtonian fluid. Such that the momentum equation (2) is therefore written as

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{K}{\rho} \frac{\partial}{\partial y} \left(-\frac{\partial u}{\partial y} \right)^n. \quad (7)$$

Introducing the following similarity variables and transformations

$$\psi(x, y) = U_w x (Re_x)^{-1/(n+1)} \cdot f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \eta = \frac{y}{x} (Re_x)^{1/(n+1)} \quad (8)$$

into Eqs. (7) and (3), we have

$$n(-f'')^{n-1} f''' - m f'^2 + \frac{2mn - m + 1}{n + 1} f f'' = 0, \quad (9)$$

$$\frac{1}{Pr} \theta'' + \frac{2mn - m + 1}{n + 1} f \theta' - \eta f' \theta = 0, \quad (10)$$

subject to the boundary conditions

$$f(0) = 0, \quad f'(0) = 1, \quad \theta(0) = 0, \quad (11a)$$

$$f'(\infty) = 0, \quad \theta(\infty) = 0, \quad (11b)$$

where $\psi(x, y)$ is the stream function which defines in usual way by $u = \partial \psi / \partial y$ and $v = -\partial \psi / \partial x$, $Re_x = \rho x^n U_w^{2-n} / K$ is the local Reynolds number, $Pr = U_w x Re_x^{-2/(n+1)} / \alpha$ is the generalized Prandtl number (refer to [16]).

To seek the global similarity solutions for this flow problem, we expand the generalized Prandtl number Pr in the following form,

$$Pr = \frac{U_w x}{\alpha} Re_x^{-2/(n+1)} = C \frac{3(n-1)}{n+1} (K/\rho)^{\frac{2}{n+1}} x^{\frac{(3m-1)(n-1)}{n+1}} / \alpha. \quad (12)$$

It is shown in Eq. (12) that, when $n = 1$ or $m = 1/3$, the Prandtl number Pr no longer contains x term such that Eqs. (9) and (10) admit the global similarity solutions. For the case $n = 1$, Eqs. (9) and (10) become the classical equations describing the boundary-layer flow and heat transfer of a Newtonian fluid over a stretching flat sheet. While for the case $m = 1/3$, the global similarity solutions are in existence for all types of fluids, i.e. the shear-thinning fluids ($n < 1$), shear-thickening fluids ($n > 1$), and the Newtonian fluid ($n = 1$).

3. Results

Using the homotopy analysis method, The analytical approximations with high accuracy are obtained in the following form:

$$f(\eta) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} \eta^i \exp(-j\lambda\eta), \quad (13)$$

$$g(\eta) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} \eta^i \exp(-j\lambda\eta), \quad (14)$$

where a_{ij} and b_{ij} are the coefficients. Due to the solution procedure on the homotopy analysis method is available in the literature [18–20], we omit the detailed process and only give some necessary messages here. The initial guesses used in the calculation are

$$f_0(\eta) = \frac{1 - \exp(-\lambda\eta)}{\lambda}, \quad \theta_0(\eta) = \exp(-\lambda\eta). \quad (15)$$

The linear operators used in the calculation are

$$\mathcal{L}_f = \frac{\partial^3 f}{\partial \eta^3} - \gamma^2 \frac{\partial f}{\partial \eta}, \quad \mathcal{L}_\theta = \frac{\partial^2 \theta}{\partial \eta^2} - \gamma^2 \theta. \quad (16)$$

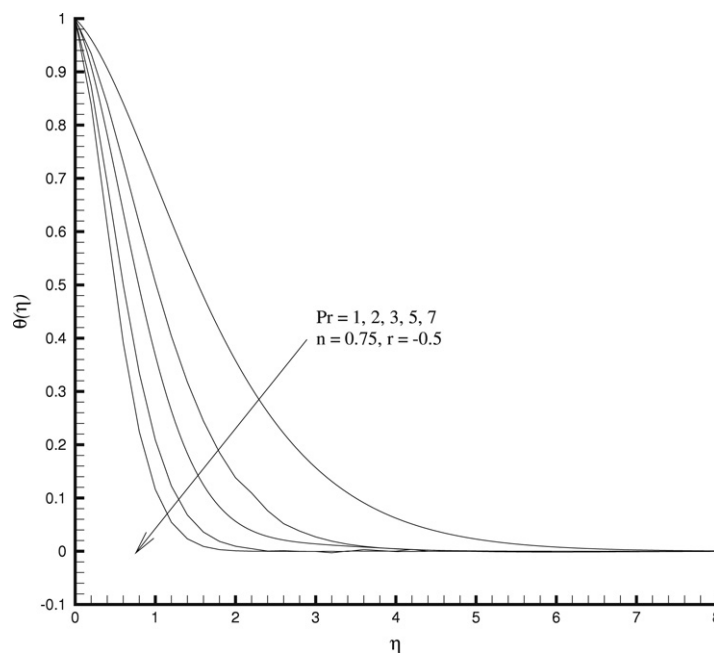
It is worth mentioning that the preceding HAM analysis is valid for arbitrary combinations of the power-law index n , the sheet stretching parameter m , and the temperature distribution parameter γ . In our calculation, all HAM approximations are obtained with the auxiliary parameters $h_f = -1/2$ and $h_\theta = -1/4$, and $\lambda = 1$. The homotopy-Padé technique [17] is employed here. We first consider the momentum boundary-layer flow with some values of n and m . As shown in Table 1,

Table 1Comparison of $f''(0)$ for various values of n with $m = 1$.

n	Present	Andersson [14]	Liao [18]	Chen [16]
0.8	−1.02853	−1.029	−1.028	−1.02919
1.0	−1.00000	−1.000		−1.00000
1.5	−0.98237	−0.981	−0.982	−0.98056
2.0	−0.97992	−0.980	−0.980	−0.98000

Table 2Comparison of $C_f Re_x^{1/(n+1)}$ for various values of n with $m = 0.5$.

n	Present	Chen [16]
1.0	−1.54074	−1.54073
1.5	−1.39578	−1.39441
1.9	−1.31970	−1.32316

**Fig. 1.** The non-dimensional temperatures distributions for some values of Pr when $n = 3/4$, $m = 1/3$, and $\gamma = -1/2$.

our [20, 20] order Homotopy-Padé [17] analytical approximations for $f''(0)$ agree well with the previous results given by Andersson [14], Chen [16], and Liao [18].

The physical interest for the non-Newtonian fluids flow is the non-dimensional skin friction C_f , defined by

$$C_f = \frac{\tau_w}{\rho U_w^2/2} = 2 Re_x^{-1/(n+1)} |f''(0)|^{n-1} f''(0), \quad (17)$$

where

$$\tau_w = \left(K \left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right)_{y=0} = \rho U_w^2 Re_x^{-1/(n+1)} |f''(0)|^{n-1} f''(0). \quad (18)$$

Table 2 shows the comparison of $|f''(0)|^{n-1} f''(0)$ for various values of n with $m = 0.5$. As can be seen, the results agree well those given in [16]. Note that Liao [18] gave his results to the boundary-layer flows for integral power-law index n and the fractional power-law index n with two different HAM techniques. However, the paper shows that present HAM technique is valid for both the two cases.

We then consider the heat transfer effect on the boundary-layer flows. As above-mentioned analysis, the global self-similarity solutions for the non-Newtonian fluid flows ($n \neq 1$) and heat transfer are valid only when $m = 1/3$. Thus we

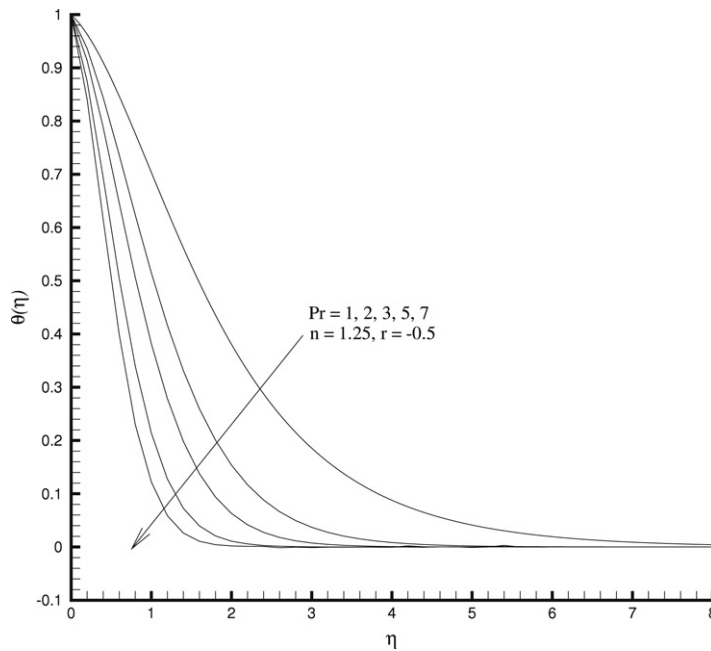


Fig. 2. The non-dimensional temperatures distributions for some values of Pr when $n = 5/4$, $m = 1/3$, and $\gamma = -1/2$.

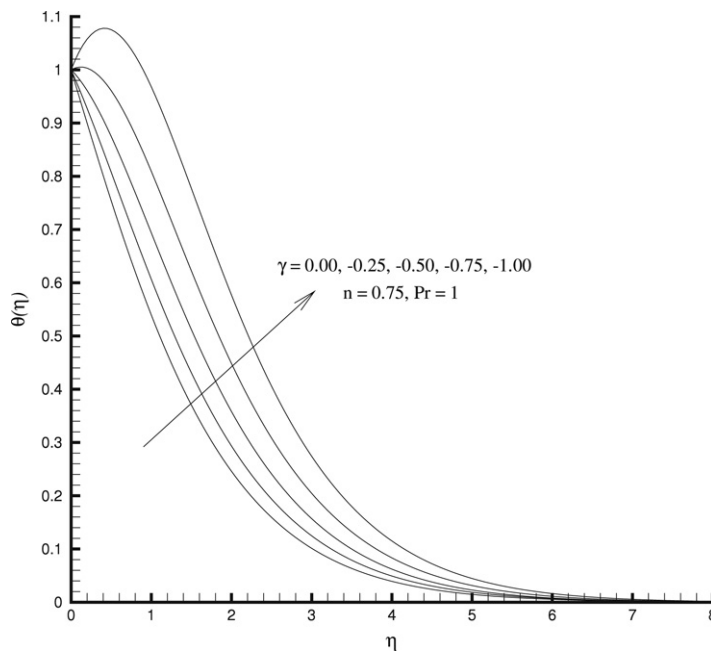


Fig. 3. The non-dimensional temperatures distributions for some values of γ when $n = 3/4$, $m = 1/3$, and $Pr = 1$.

only investigate this case here. As shown in Figs. 1 and 2, the non-dimensional temperature profiles decreases as the Prandtl number Pr increases for both the shear-thinning fluids ($n < 1$) and shear-thickening fluids ($n > 1$) for $r = -1/2$. However, the non-dimensional temperature profiles increases as the temperature distribution parameter r decreases for both the shear-thinning fluids ($n < 1$) and shear-thickening fluids ($n > 1$), as shown in Figs. 3 and 4.

Note that the generalized Prandtl number Pr defined in [16] contains x so that his heat transfer analysis is local. These kinds of results are only valid for large x or small x . Different from Chen's analysis [16], the present analysis is valid for all x . To the best of our knowledge, this problem has not been reported.

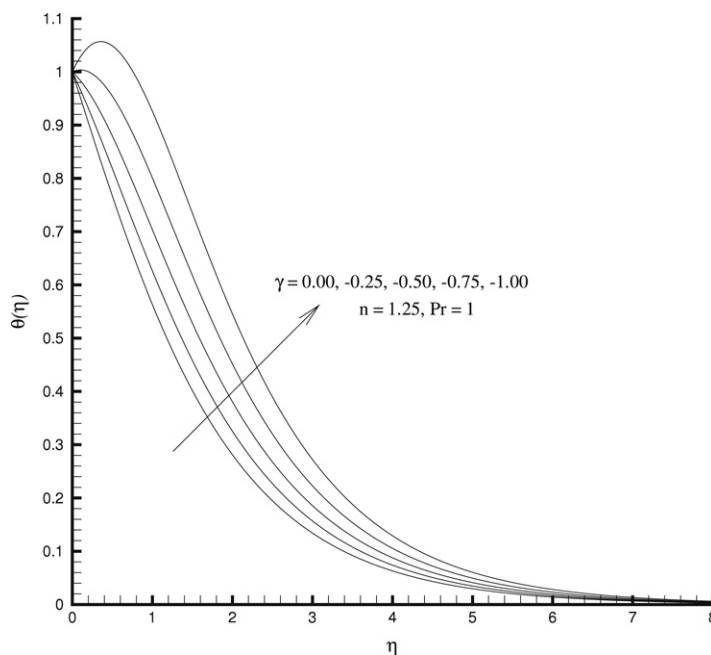


Fig. 4. The non-dimensional temperatures distributions for some values of γ when $n = 5/4$, $m = 1/3$, and $Pr = 1$.

4. Concluding remarks

In this paper, the boundary-layer flow and heat transfer of power-law non-Newtonian fluids over a stretching sheet with the sheet velocity distribution of the form $U(x) = Cx^m$ and the wall temperature distribution of the form $T_w = T_\infty + Ax^\gamma$ are studied. The global self-similarity equations for the flow problem are formulated. The accurate analytical approximations are then obtained with the help of the homotopy analysis method.

Most previous studies on the boundary-layer flow and heat transfer are based on the linear plate stretching velocity ($U_w = ax$), where the global self-similarity solutions are valid. The present analysis provides a deep insight into the boundary layer flow and the heat transfer for the power-law plate stretching velocity and the power-law temperature distribution. Similar ideas are expected to apply to other nonlinear problems about the non-Newtonian fluid flows in a similar way.

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